1.9 Nested Qualifiers

- A logical expression with more than one quantifier that binds different variables in the same predicate has <u>nested qualifiers</u>.

- A logical expression is a proposition if all the variables are bound.

$\forall x \exists y\, P(x,y)$ x and y are both bound
$\forall x\, P(x,y)$ x is bound and y is free
$\exists y \exists z\, T(x,y,z)$ y and z are bound. x is free

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$M(x,y)$ = x sent an email to y

$\forall x \forall y\, M(x,y) \Leftrightarrow$ "Everyone sent an email to everyone"
- The statement includes the case that $x=y$, so if $\forall x \forall y\, M(x,y)$ is true, then everyone sent an email to everyone else and everyone sent an email to themselves.

$\exists x \exists y\, M(x,y) \leftrightarrow$ "There is a person who sent an email to someone"
- This statement is true even if the only email sent was one person emailing themselves.

$\exists x \forall y\, M(x,y) \Leftrightarrow$ "There is a person who sent an email to everyone"

$\forall x \exists y\, M(x,y) \Leftrightarrow$ "Everyone sent an email to someone."

Player	Action	Goal
Existential Player	Selects values for existentially bound variables	Tries to make the expression true
Universal Player	Selects values for universally bound variables	Tries to make the expression false

Examples

Domain: set of all integers

Ex 1 $\forall x \exists y\, (x+y=0)$

- The universal player selects value of x
- The existential player wants to make the statement true
- The existential player can always select y to be $-x$ to make the statement true
- TRUE

Ex 2 $\exists x \forall y\, (x+y=0)$

- The existential player selects value of x
- The universal player wants the statement to be false
- The universal player can always select a value of y to make the statement false
- FALSE

$\forall x \forall y \exists z\, (x + y = z) \longleftrightarrow$ "For every x and every y, there is a z, such that $x+y=z$"

- True

$\exists z \forall x \forall y\, (x+y=z) \longleftrightarrow$ "There is a z, such that for every x and every y, $x+y=z$"

- False
- Z is chosen first
- Once Z is chosen, x and y can vary to make $x+y=z$ be false.

$\exists x \forall y \forall z\, P(x,y,z) \equiv \exists x \forall z \forall y\, P(x,y,z)$

$\exists x \exists y \forall z\, P(x,y,z) \equiv \exists y \exists x \forall z\, P(x,y,z)$

De Morgans Law

$\neg \forall x \forall y\, P(x,y) \equiv \exists x \exists y\, \neg P(x,y)$

$\neg \forall x \exists y\, P(x,y) \equiv \exists x \forall y\, \neg P(x,y)$

$\neg \exists x \forall y\, P(x,y) \equiv \forall x \exists y\, \neg P(x,y)$

$\neg \exists x \exists y\, P(x,y) \equiv \forall x \forall y\, \neg P(x,y)$

1.10 More nested quantified statements

In the same case as above how would you define a predicate that indicates whether x sent an email to y.

↳ $\forall x \forall y \left[(x \neq y) \rightarrow M(x,y) \right]$

"Everyone sent an email to someone else" $\equiv \forall x \exists y [(x \neq y) \wedge M(x,y)]$

$L(x)$: x was late to the meeting

$\exists x \left(L(x) \wedge \forall y((x \neq y) \rightarrow \neg L(y))\right)$ $\equiv$ "Exactly one person was late to the meeting"

Moving quantifiers in logical statements.

$$A \wedge \forall y P(y,*) \equiv \forall y (A \wedge P(y,*))$$

- This equivalence is an example of the null qualification rule.
  - The $*$ in $P(y,*)$ indicates a predicate P can depend on other variables besides y.

- For biconditional statements you CANNOT move quantifiers around.

1.11 Logical Reasoning

- An argument is a sequence of propositions called hypotheses, followed by a final proposition, called the conclusion.

- An argument is valid if the conclusion is true for every truth assignment to the variables that causes all of the hypotheses to be true.

hypothesis notation:

$$\boxed{\begin{array}{l} h_1 \\ h_2 \\ \cdots \\ h_n \\ \hline \therefore c \end{array}}$$

$\therefore$ = "therefore"

- Argument is valid whenever $(h_1 \wedge h_2 \wedge \ldots \wedge h_n) \rightarrow c$

Example

Argument

$$\boxed{\begin{array}{l} p \rightarrow q \\ p \vee q \\ \hline \therefore q \end{array}}$$

Truth Table

p	q	p → q	p ∨ q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

1) In which rows are all the hypotheses true?

Ans: Rows 1 and 3

2) Is the conclusion true in all of the rows where both the hypotheses are true?

Ans: yes

3) Is the argument valid?

Ans: yes

★ The argument is only invalid if all the hypotheses are True and the conclusion is false

Argument Form A (Valid)

$$\boxed{\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}}$$

Argument of B (Invalid)

$$\boxed{\begin{array}{l} \neg p \\ p \rightarrow q \\ \hline \therefore \neg q \end{array}}$$